MATH 2040 Lecture 8 (Oct 3, 2016)

§ Characterization for diagonalizability (textbook 5.2)

Theorem: Let $T: V \to V$ linear, $\dim V = n < +\infty$. / $\mathbb{F}$.

Suppose the char. poly. of T, $f(t)$, splits over $\mathbb{F}$, i.e.

$$f(t) := \det(T - tI)$$

$$= (-1)^n (t-\lambda_1)^{m_1} \cdots\cdots (t-\lambda_k)^{m_k}$$

distinct eigenvalues : $\lambda_1, \ldots\ldots, \lambda_k$

(algebraic) multiplicity : $m_1, \ldots\ldots, m_k$

Then, the following are true:

(a) T is diagonalizable $\Leftrightarrow$ $\underbrace{\dim E_{\lambda_i}}_{\text{geometric multiplicity}} = m_i$ for each i.

(b) If T is diagonalizable,

and β_i is a basis for E_{λ_i}

then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is eigenbasis for T (of V)

Proof: (a) ($\Rightarrow$) last time. (a) ($\Leftarrow$) follows from (b)

(b) It suffices to prove that

Lemma: $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ linearly indep.

$$\beta = \overset{E_{\lambda_1}}{\beta_1} \cup \beta_2 \cup \cdots \cup \overset{E_{\lambda_k}}{\beta_k}$$

$$\beta_1 = \{\vec{v}_{11}, \vec{v}_{12}, \dots, \vec{v}_{1n_1}\} \qquad \beta_k = \{\vec{v}_{k1}, \vec{v}_{k2}, \dots, \vec{v}_{kn_k}\}$$

$$\dim E_{\lambda_1} = n_1 \qquad \dim E_{\lambda_k} = n_k$$

Assume $\exists$ linear combination

$$\underbrace{(c_{11}\vec{v}_{11} + c_{12}\vec{v}_{12} + \cdots + c_{1n_1}\vec{v}_{1n_1})}_{= \vec{v}_1 \in E_{\lambda_1}} + \underbrace{\cdots\cdots}_{= \vec{v}_2 \in E_{\lambda_2}} + \underbrace{(c_{k1}\vec{v}_{k1} + c_{k2}\vec{v}_{k2} + \cdots + c_{kn_k}\vec{v}_{kn_k})}_{= \vec{v}_k \in E_{\lambda_k}} = \vec{0}$$

(Recall: Take $\vec{v}_i \in E_{\lambda_i} \Rightarrow \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ linearly indep.)

distinct eigenspace

$\Rightarrow$ Each $\vec{v}_i = c_{i1}\vec{v}_{i1} + c_{i2}\vec{v}_{i2} + \cdots + c_{in_i}\vec{v}_{in_i} = \vec{0}$ for any i.

β_i basis $\Rightarrow c_{i1} = c_{i2} = \cdots = c_{in_i} = 0 \Rightarrow \beta$ lin. indep.

$\square$

(a) ($\Leftarrow$) Assume $\dim E_{\lambda_i} = m_i \Rightarrow \beta$ is a basis for V

$\because \# \beta = \# \beta_1 + \# \beta_2 + \cdots + \# \beta_k = m_1 + m_2 + \cdots + m_k = n$

$f(t)$ splits

$\square$

Rephrase the theorem as

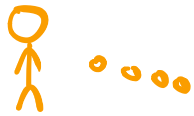
$$E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k} \overset{T \text{ diagonalizable}}{=} V$$

$\subseteq$

general T

Application: Taking A^k for large k.

Eg. $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ Q: What is A^{10000} ?



For $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ diagonal.

$$\Rightarrow D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$$ for any k.

Idea: If A is diagonalizable,

then $Q^{-1} A Q = D$ ← diagonal

$$\Rightarrow (Q^{-1} A Q)^k = D^k \quad \leftarrow \text{easy!}$$

$$\| $$

$$(Q^{-1} A Q)(Q^{-1} A Q) \cdots\cdots (Q^{-1} A Q)$$

$$\|$$

$$Q^{-1} \underbrace{A A A \cdots A}_{k\text{-times}} Q$$

ie. $Q^{-1} A^k Q = D^k \Rightarrow A^k = Q D^k Q^{-1}$

Back to this example $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$

~~2nd suff. test~~ $\because$ not symmetric

$f(t) = \det(A - tI) = t^2 - 3t + 2 = 0$

$\Rightarrow \quad \lambda_1 = 1 \quad , \quad \lambda_2 = 2$ $\qquad$ 1^{st} suff. test ✓

$\Downarrow$

diagonalizable

$\because \exists$ 2 distinct e.values

for 2×2 matrix A

compute eigenspaces:

$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \quad , \quad E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\Rightarrow \quad Q = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

$\Rightarrow \quad A^{10000} = Q \begin{pmatrix} 1^{10000} & 0 \\ 0 & 2^{1000} \end{pmatrix} Q^{-1} = \text{ans.}$ ✱

§ Invariant Subspaces

Idea: Given a vector space V

and a linear $T : V \to V$

is there a "subspace" includes both data V & T ?

Def$^{\underline{n}}$: A subspace $W \subseteq V$ is T-invariant

if $T(W) \subseteq W$

Note: $\nexists$ $T(\vec{w}) \neq \vec{w}$ even $T(W) = W$

$\vec{w} \in W$

Examples: (0) trivial: $\{\vec{0}\}$ or V

(1) $N(T)$ & $R(T)$

Proof: Pick any $\vec{w} \in R(T)$

$$T\vec{w} \overset{?}{\in} R(T) := \{T\vec{v} \in V : \vec{v} \in V\}$$

✓ by def^{n}.

(2) E_λ is T-invariant.

Proof: Pick $\vec{w} \in E_\lambda \Rightarrow T\vec{w} = \lambda\vec{w}$

Check: $T\vec{w} \overset{?}{\in} E_\lambda$

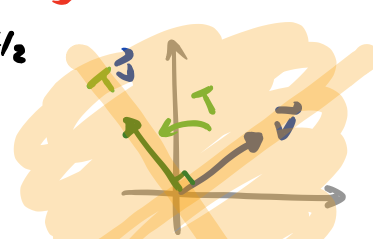
$$T(\underbrace{T\vec{w}}_{=\lambda\vec{w}}) \overset{?}{=} \lambda(\underbrace{T\vec{w}}_{=\lambda\vec{w}}) \quad ✓$$

Geometrically:

Q: Given $\vec{v} \in V$, what is the smallest T-inv. subspace containing $\vec{v}$?

E.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$ rot. by $\pi/2$

$\vec{v} \neq \vec{0} \rightsquigarrow W = \mathbb{R}^2$



In general, if W is the smallest T-inv. subspace containing $\vec{v}$

$$W = \left\{ \underbrace{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots, T^k\vec{v}, \ldots}_{} \ \& \text{ all linear combinations of} \right\}$$

i.e. $W := \operatorname{span}\{ \underbrace{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots, T^k\vec{v}, \ldots}_{\infty\text{'ly many vectors}} \} \leq V$

<u>Defⁿ</u>: This is T-cyclic subspace generated by $\vec{v}$

Why consider all this?

$\because$ We can restrict to a smaller subspace?

$$\begin{array}{ccccc} T & : & V & \longrightarrow & V \\ & & \cup\!\text{I} & & \cup\!\text{I} \\ T|_W & : & W & \longrightarrow & W \\ & & & & \cup\!\text{I} \\ & & & & T(W) \end{array}$$

T-invariant $\Rightarrow$ $T(W)$

restriction of T to W

<u>Morale</u>: understand $T|_W$ $\rightsquigarrow$ understand T (more)

(better hopefully)

e.g. $t-1 \mid (t-1)^2(t+2)$

<u>Lemma</u>:

$$\left.\begin{array}{lcl} T : V \to V & \rightsquigarrow & f_T(t) \\ \quad\ \cup\!\text{I} \quad\ \cup\!\text{I} & & \text{w/ deg} \\ T|_W : W \to W & \rightsquigarrow & f_{T_W}(t) \end{array}\right\} \Rightarrow f_{T_W}(t) \mid f_T(t)$$

divides.